Hollow sphere under internal pressure

A homogeneous hollow sphere (internal radius a, external radius b) is made of an elastic perfectly plastic material (linear elasticity, characterized by the Young's modulus E and the Poisson's ration ν ; yield stress σ_y). The yield criterion is von Mises.

Starting from an initial stress free state, the internal pressure rises from 0 to p (Fig.1a).

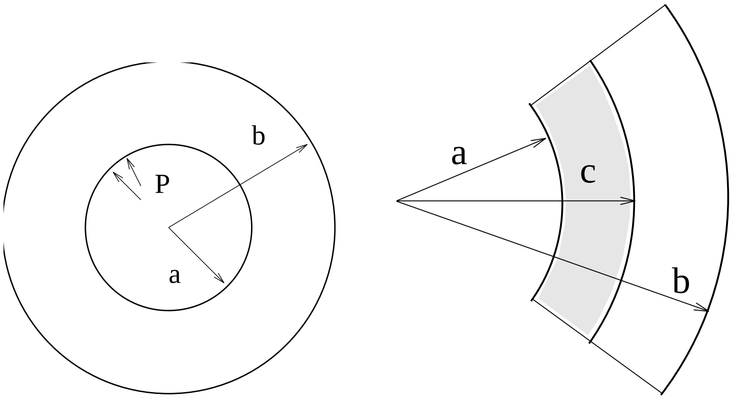

FIG. 1 – (a) Geometry and applied load, (b) illustration of the plastic zone growing from the internal surface

Elastic analysis

Write the solution, in terms of stress and displacement, for an purely elastic behaviour.

Since the body's geometry and the loading present a spherical symmetry, and since the material's behaviour is isotropic, the solution can be built in a system of spherical coordinates, (r, θ, ϕ). The displacement field, the stress field and the strain field write respectively :

$$\begin{array}{lll} u_r = h(r) & u_\theta = u_\phi = 0 & \\ \sigma_r = f_1(r) & \sigma_{\theta\theta} = \sigma_\phi = g_1(r) & \sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\phi\theta} = 0 \\ \varepsilon_r = f_2(r) & \varepsilon_{\theta\theta} = \varepsilon_\phi = g_2(r) & \varepsilon_{r\theta} = \varepsilon_{r\phi} = \varepsilon_{\phi\theta} = 0 \end{array}$$

The equilibrium, the static and kinematic boundary conditions can be expressed as :

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \qquad (1)$$

$$\sigma_r(r = a) = -p \qquad \sigma_r(r = b) = 0 \qquad (2)$$

$$\varepsilon_r = \frac{du_r}{dr}, \quad \varepsilon_\theta = \frac{u_r}{r} \qquad (3)$$

The elastic constitutive equations write :

$$E\varepsilon_r = (\sigma_r - 2\nu\sigma_\theta) \quad E\varepsilon_\theta = \sigma_\theta(1-\nu) - \nu\sigma_r \qquad (4)$$

After replacing strain by its expression as a function of displacement, it comes (using Lamé's coefficient $\lambda = E\nu/(1-2\nu)/(1+\nu)$) :

$$\sigma_r = \frac{\lambda}{\nu}\left((1-\nu)\frac{du_r}{dr} + 2\nu\frac{u_r}{r}\right) \quad \sigma_\theta = \frac{\lambda}{\nu}\left(\nu\frac{du_r}{dr} + \frac{u_r}{r}\right) \qquad (5)$$

These two equations can be introduced in the equilibrium equation, leading to :

$$\frac{d^2u_r}{dr^2} + \frac{2}{r}\frac{du_r}{dr} - \frac{2}{r^2}u_r = 0 \tag{6}$$

that is :

$$\left(\frac{1}{r^2}\left(r^2 u_r\right)_{,r}\right)_{,r} = 0 \tag{7}$$

The solution of this equation is :

$$u_r = C_1 r + \frac{C_2}{r^2} \tag{8}$$

By replacing u_r in the preceding expressions, it comes :

$$\sigma_r = \frac{\lambda}{\nu}\left((1+\nu)C_1 - 2(1-2\nu)\frac{C_2}{r^3}\right) \tag{9}$$

$$\sigma_\theta = \frac{\lambda}{\nu}\left((1+\nu)C_1 + (1-2\nu)\frac{C_2}{r^3}\right) \tag{10}$$

The integration constants C_1 and C_2 can be determined from the boundary conditions :

$$\sigma_r(r=b) = 0 \quad \Rightarrow \; C_2 = \frac{1+\nu}{2(1-2\nu)} b^3 C_1 \tag{11}$$

$$\sigma_r(r=a) = -p \Rightarrow \; C_1 = \frac{1-2\nu}{E}\frac{a^3}{b^3-a^3}\,p \tag{12}$$

So that :

$$\sigma_r = -\frac{a^3}{b^3-a^3}\left(\frac{b^3}{r^3} - 1\right)p \tag{13}$$

$$\sigma_\theta = \sigma_\phi = \frac{a^3}{b^3-a^3}\left(\frac{b^3}{2r^3} + 1\right)p \tag{14}$$

$$u_r = \frac{a^3}{b^3-a^3}\left((1-2\nu)r + (1+\nu)\frac{b^3}{2r^2}\right)\frac{p}{E} \tag{15}$$

Determine the pressure P_e that corresponds to the onset of plastic flow, for von Mises and Tresca criteria.

Both criteria are insensitive to hydrostatic pressure. They resulting value is then not affected if a spherical tensor is added. In the present case, if the diagonal (σ_θ, σ_θ, σ_θ) is subtracted from the actual tensor the new tensor is simply uniaxial, the only non zero component (11) being $\sigma_r - \sigma_\theta$, which is then the value of both von Mises and Tresca criteria. Using the solution of the preceding question, this value can be evaluated in the elastic regime, as :

$$\sigma_\theta - \sigma_r = \frac{3}{2}\frac{a^3}{b^3-a^3}\frac{b^3}{r^3}p \tag{16}$$

The plastic yield is reached when $(\sigma_\theta - \sigma_r)$, increasing function of p, is equal to σ_y, the yield limit in onedimensional tension. The initial plastic zone is then located in $r = a$, and the value of the pressure P_e is :

$$P_e = \frac{2}{3}\left(1 - \frac{a^3}{b^3}\right)\sigma_y \tag{17}$$

Elasto-plastic solution

Find the stress and displacement fields in the elastoplastic regime. Check that the plastic zone expands from the internal radius of the sphere (Fig.1b). Determine the relation between the radius of the plastic zone c and the pressure p. Determine the limit pressure that produces failure by excessive deformation, P_p.

When the internal pressure p becomes larger than P_e, a natural assumption is that the plastic zone takes a region $a < r < c$, with c a increasing function of p, meanwhile the region $c < r < b$ remains in the elastic regime. The component σ_r is continuous at the boundary between the elastic and the plastic zone, and, at this point, the value of the absolute value of the normal stress is nothing but the stress that corresponds to the onset of plastic flow for a hollow sphere of external radius b and internal radius c, that is :

$$\sigma_r(c) = -\frac{2}{3}\left(1-\frac{c^3}{b^3}\right)\sigma_y \tag{18}$$

The stress components in the elastic zone are then given by equations (15) where a is replaced by c and $-p$ by $-\sigma_r(c)$. It comes then :

$$\sigma_r = -\frac{2}{3}\frac{c^3}{b^3}\left(\frac{b^3}{r^3}-1\right)\sigma_y \tag{19}$$

$$\sigma_\theta = \frac{2}{3}\frac{c^3}{b^3}\left(1+\frac{b^3}{2r^3}\right)\sigma_y \tag{20}$$

$$u_r = \frac{2}{3E}\frac{c^3}{b^3}\left((1-2\nu)r+(1+\nu)\frac{b^3}{2r^2}\right)\sigma_y \tag{21}$$

Let us study the plastic zone, $a < r < c$. The elastic equations are not valid any more, and, since the strain remains a priori undetermined, the only equations available to calculate the stress components are, for any point in the plastic zone :

- the equilibrium equations

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \tag{22}$$

- the plasticity criterion :

$$\sigma_r - \sigma_\theta = -\sigma_y \tag{23}$$

So that

$$\frac{d\sigma_r}{dr} - \frac{2}{r}\sigma_y = 0 \qquad \sigma_r = 2\sigma_y \ln(r) + C_3 \tag{24}$$

The integration constant C_3 can be determined by the condition at $r = c$, by using the continuity of the component σ_r :

$$2\sigma_y \ln(c) + C_3 = -\frac{2}{3}\left(1-\frac{c^3}{b^3}\right)\sigma_y \tag{25}$$

In the plastic zone, we have in turn :

$$\sigma_r = -\frac{2}{3}\sigma_y\left(1+3\ln(\frac{c}{r})-\frac{c^3}{b^3}\right) \tag{26}$$

$$\sigma_\theta = \frac{2}{3}\sigma_y\left(\frac{1}{2}-3\ln(\frac{c}{r})+\frac{c^3}{b^3}\right) \tag{27}$$

These stress components depend on the parameter c, the evolution of which has be determined as a function of p. In the plastic zone, for $r = a$, $\sigma_r(r = a) = -p$, and :

$$p = \frac{2}{3}\sigma_y\left(1+3\ln\left(\frac{c}{a}\right)-\frac{c^3}{b^3}\right) \tag{28}$$

Since the displacements in the sphere remain small, a et b are supposed to be constant. Taking the derivative of p with respect to c :

$$\frac{dp}{dc} = \frac{2\sigma_y}{c}\left(1-\frac{c^3}{b^3}\right) \tag{29}$$

This term is always positive. The radius c of the plastic zone is an increasing function of p. This result is consistent with the initial assumption that the plastic zone grows from the internal radius of the sphere. The external radius is reached by the plastic zone for the limit pressure P_p :

$$P_p = 2\sigma_y \ln\left(\frac{b}{a}\right) \tag{30}$$

Determine the plastic strain and the plastic strain rate.

If a solution for the displacement field can be obtained from the stress field determined at the previous question, by using the constitutive equations, and that this field is compatible with the kinematic boundary conditions, the solution is unique.

Due to the spherical symmetry condition, the only non zero component of the displacement is radial. As plasticity does not produce any volume change, the volume variation is due to the elastic strain only :

$$\varepsilon_r + 2\varepsilon_\theta = \frac{1-2\nu}{E}(\sigma_r + 2\sigma_\theta) \tag{31}$$

This allows us to obtain the expression of the radial component of the displacement :

$$\frac{du_r}{dr} + 2\frac{u_r}{r} = -\frac{2(1-2\nu)}{E}\sigma_y\left(3\ln\left(\frac{c}{r}\right) - \frac{c^3}{b^3}\right) \tag{32}$$

$$u_r = \frac{C_4}{r^2} - \frac{2(1-2\nu)}{E} r\sigma_y\left(\ln\left(\frac{c}{r}\right) + \frac{1}{3}\left(1-\frac{c^3}{b^3}\right)\right) \tag{33}$$

Since the radial displacement is continuous by crossing the elastic–plastic boundary at $r = c$, the integration constant C_4 can be determined :

$$C_4 = (1-\nu)\frac{\sigma_y}{E}c^3 \tag{34}$$

So that, in the plastic zone :

$$u_r = \frac{\sigma_y}{E} r\left((1-\nu)\frac{c^3}{r^3} - \frac{2}{3}(1-2\nu)\left(1+3\ln\left(\frac{c}{r}\right) - \frac{c^3}{b^3}\right)\right) \tag{35}$$

The strain in the plastic zone can be calculated by using this expression. The plastic strain can then be obtained by subtracting the elastic strain (known from the stress values) to the total strain. The only non zero components are :

$$\varepsilon_r^p = \frac{2\sigma_y}{E}(1-\nu)\left(1-\frac{c^3}{r^3}\right) \tag{36}$$

$$\varepsilon_\theta^p = \varepsilon_\phi^p = -\frac{\sigma_y}{E}\left((1-\nu)(1-\frac{c^3}{r^3})\right) \tag{37}$$

As c is an increasing function of p and that the applied loading is for increasing values of p, c can be chosen as a loading parameter. The strain rate tensor's type is like pure onedimensional compression :

$$\dot{\varepsilon}_r^p = \frac{d\varepsilon_r^p}{dc} = -\frac{6\sigma_y}{E}(1-\nu)\frac{c^2}{r^3} < 0 \tag{38}$$

$$\dot{\varepsilon}_\theta^p = \dot{\varepsilon}_\phi^p = -\frac{1}{2}\dot{\varepsilon}_r^p \tag{39}$$

It can be checked that the plastic strain value is zero at $r = c$. Its maximum is found for $r = a$:

$$\varepsilon_r^p = \frac{2\sigma_y}{E}(1-\nu)\left(1-\frac{c^3}{a^3}\right) \tag{40}$$

The maximum value when the pression P_p is reached is then :

$$\varepsilon_r^p = \frac{2\sigma_y}{E}(1-\nu)\left(1-\frac{b^3}{a^3}\right) \tag{41}$$

What happens for the following loading path : $0 \to p_m\,(p_m > P_e) \to 0 \to p_m\,(p_m > P_e)$?

If a hollow sphere has been submitted to an internal pressure $P_m > P_e$, and that the pressure comes back to zero ($p = 0$), residual stresses will develop during the unloading branch. The first idea is then to assume an elastic unloading, and to calculate the residual field as the difference between the stresses at maximum load ($p = P_m$) and the elastic solution obtained for the opposite loading ($p = -P_m$). The residual stress tensor $\underset{\sim}{\sigma}^R$ is then :

- in the plastic zone ($a \leqslant r \leqslant c$) :

$$\sigma_r^R = -\frac{2}{3}\sigma_y\left(1+3\ln\left(\frac{c}{r}\right)-\frac{c^3}{b^3}\right)+\frac{a^3}{b^3-a^3}\left(\frac{b^3}{r^3}-1\right)P_m \tag{42}$$

$$\sigma_\theta^R = \sigma_\phi^r = \frac{2}{3}\sigma_y\left(\frac{1}{2}-3\ln\left(\frac{c}{r}\right)+\frac{c^3}{b^3}\right)-\frac{a^3}{b^3-a^3}\left(\frac{b^3}{2r^3}+1\right)P_m \tag{43}$$

- in the elastic zone ($c \leqslant r \leqslant b$) :

$$\sigma_r^R = -\frac{2}{3}\frac{c^3}{b^3}\left(\frac{b^3}{r^3}-1\right)\sigma_y+\frac{a^3}{b^3-a^3}\left(\frac{b^3}{r^3}-1\right)P_m \tag{44}$$

$$\sigma_\theta^r = \sigma_\phi^r = \frac{2}{3}\frac{c^3}{b^3}\left(1+\frac{b^3}{2r^3}\right)\sigma_y-\frac{a^3}{b^3-a^3}\left(\frac{b^3}{2r^3}+1\right)P_m \tag{45}$$

The equations (45) are valid only if there is no reverse plastic flow during unloading. In fact, plastic flow might happen if the elastic domain is crossed, and that a point such as $\sigma_\theta - \sigma_r = -\sigma_y$ is reached. That can be the case if the maximum pressure P_m is larger than $2P_e$. Since P_m itself must be smaller than P_p (or the sphere would fail during the pressure increase), the previous condition is replaced by $P_p > 2P_e$. This last inequality provides a condition on the geometry of the sphere. Figure 2 illustrates the fact that P_p is larger than $2P_e$ iff the ratio (a/b) is smaller than a critical value x, solution of the equation

$$\frac{4}{3}(1-x^3)+2\ln(x) = 0 \tag{46}$$

that is

$$a/b < x \simeq 0.59 \tag{47}$$

If there is no reverse plastic flow during the unloading, the structure is called “adapted”. This is a safe operation mode, which is classically used in the pressure vessels. Before operation, these vessels are submitted to a preloading with a pressure larger than the pressure in subsequent operations. The result will be a compressive in-plane stress component at the internal radius.

On the contrary, if plastic flow takes place, cyclic plastic strain will be present at the internal surface, leading to a plastic fatigue regime, and to the failure of the structure after some cycling period.

Figure 3 illustrates the variations of the various components of the stress tensor at maximal load and after unloading, for the case $(a/b) = 0.75$.

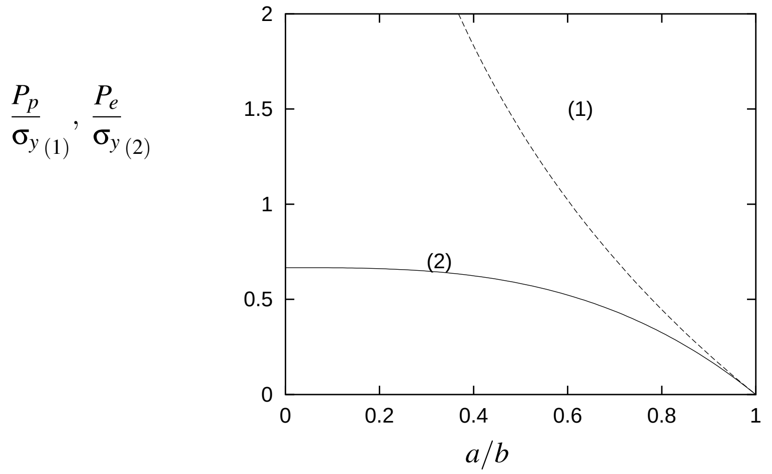

FIG. 2 – Variation of P_e and P_p versus a/b

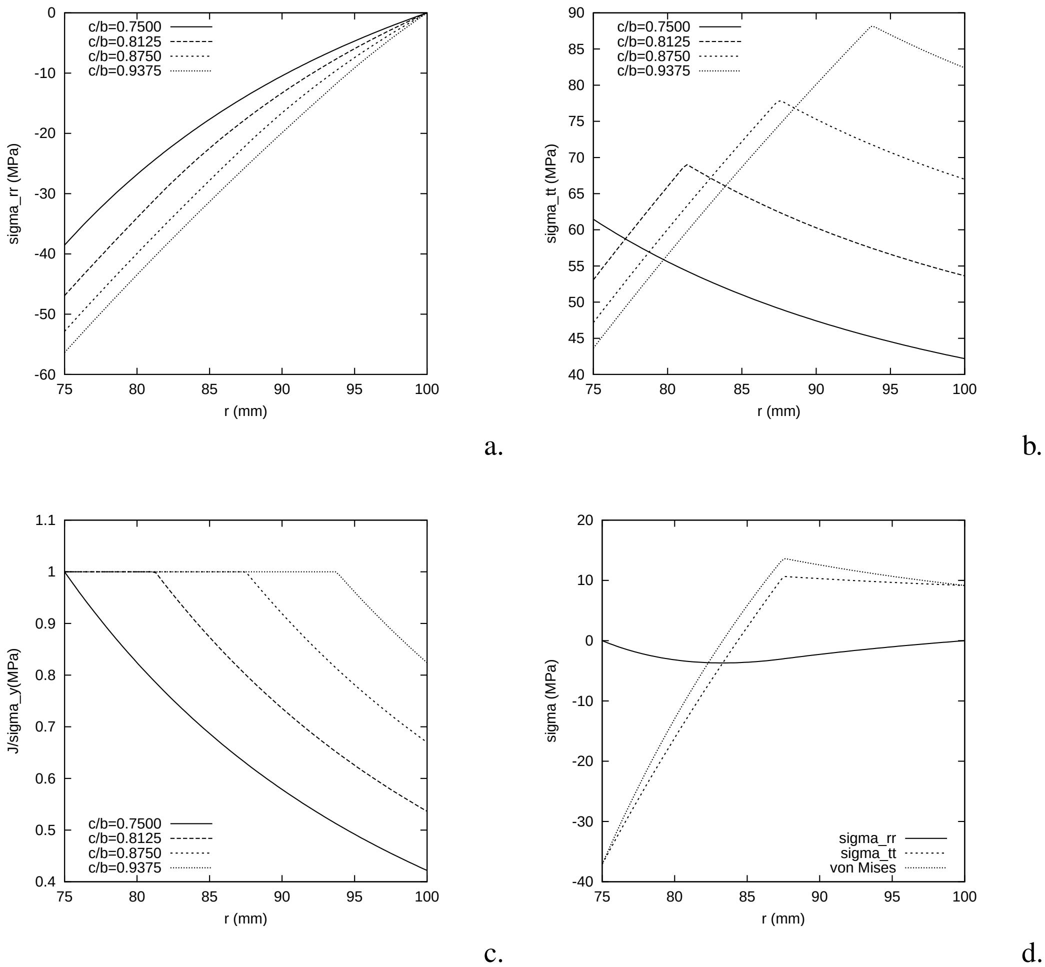

FIG. 3 – (a), (b) Stress profiles in the sphere for various external pressure values (c) growth of the plastic zone, (d) residual stresses after unloading