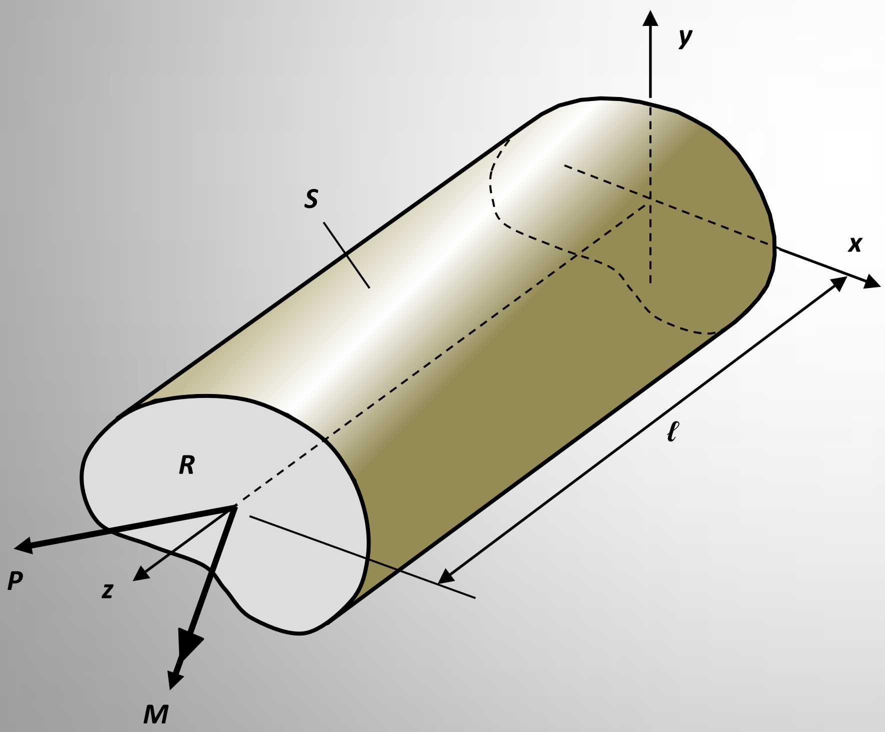
Chapter 9
Extension, Torsion and Flexure of Elastic Cylinders

Prismatic Bar Subjected to End Loadings



Semi-Inverse Method

Assume: $\sigma_x = \sigma_y = \tau_{xy} = 0$

Equilibrium Equations $\Rightarrow$

$$\frac{\partial \tau_{xz}}{\partial z} = \frac{\partial \tau_{yz}}{\partial z} = 0$$

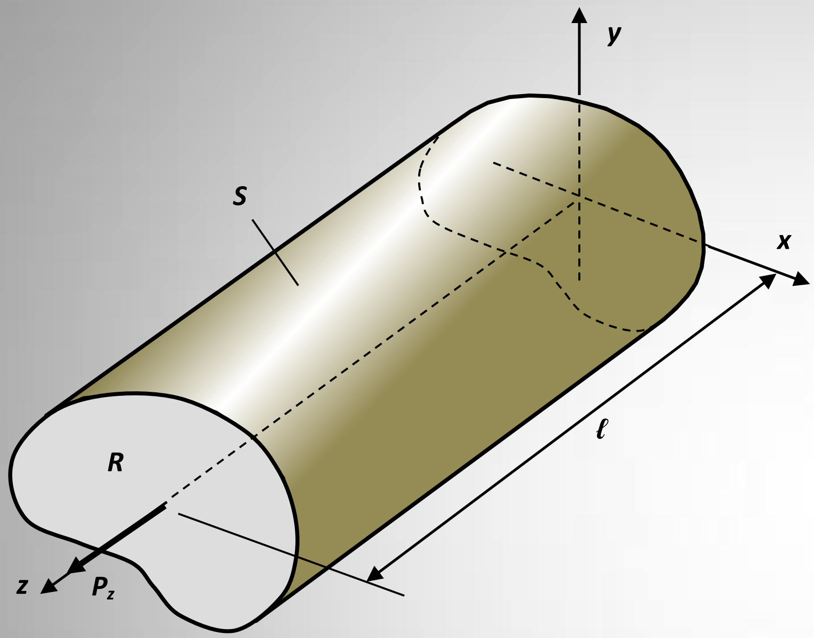
Compatibilty Equations $\Rightarrow$

$$\frac{\partial^2 \sigma_z}{\partial x^2} = \frac{\partial^2 \sigma_z}{\partial y^2} = \frac{\partial^2 \sigma_z}{\partial z^2} = \frac{\partial^2 \sigma_z}{\partial x \partial y} = 0$$

Integrating $\Rightarrow$

$$\sigma_z = C_1 x + C_2 y + C_3 z + C_4 xz + C_5 yz + C_6$$

Extension of Cylinders



Assumptions

- Load P_z is applied at centroid of cross-section so no bending effects
- Using Saint-Venant Principle, exact end tractions are replaced by statically equivalent uniform loading
- Thus assume stress σ_z is uniform over any cross-section throughout the solid

$$\Rightarrow \sigma_z = \frac{P_z}{A}, \ \tau_{xz} = \tau_{yz} = 0$$

and $\sigma_x = \sigma_y = \tau_{xy} = 0$

Using stress results into Hooke's law and combining with the strain-displacement relations gives

$$\frac{\partial u}{\partial x} = -\frac{\nu P_z}{AE}, \ \frac{\partial v}{\partial y} = -\frac{\nu P_z}{AE}, \ \frac{\partial w}{\partial z} = \frac{P_z}{AE}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

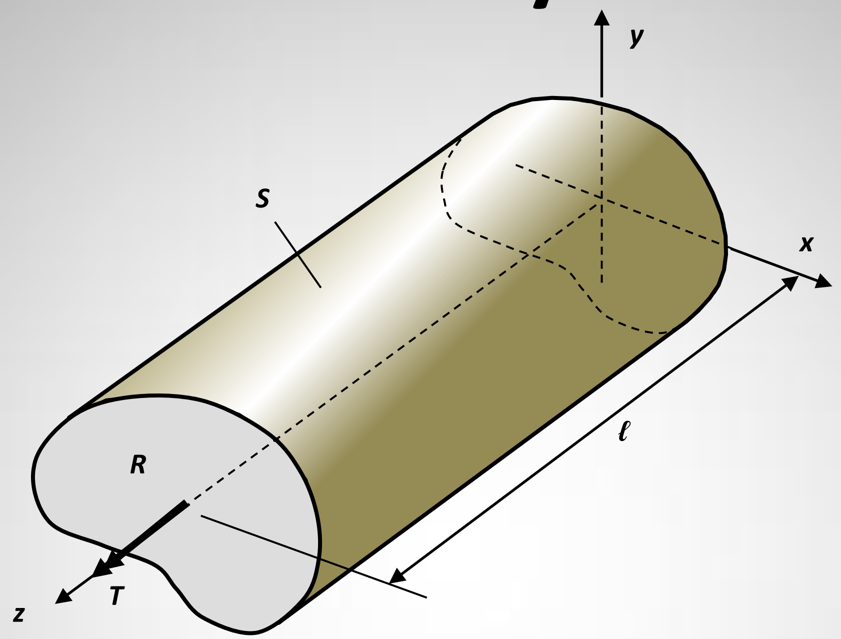
Integrating and dropping rigid-body motion terms such that displacements vanish at origin $\Rightarrow$

$$u = -\frac{\nu P_z}{AE} x$$

$$v = -\frac{\nu P_z}{AE} y$$

$$w = \frac{P_z}{AE} z$$

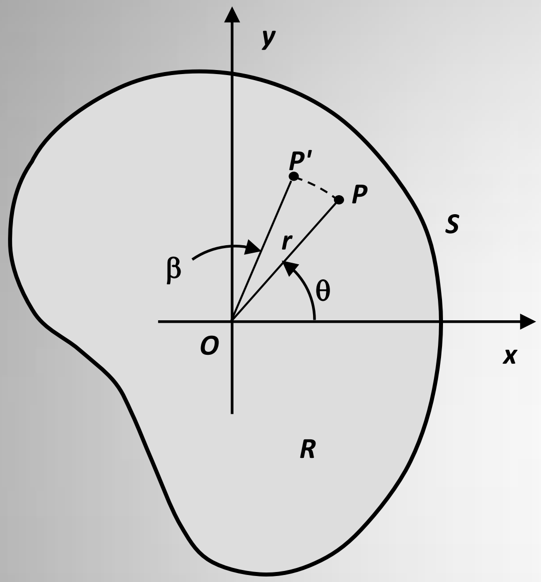
Torsion of Cylinders



Guided by Observations from Mechanics of Materials

- projection of each section on *x,y*-plane rotates as rigid-body about central axis
- amount of projected section rotation is linear function of axial coordinate
- plane cross-sections will not remain plane after deformation thus leading to a warping displacement

Torsional Deformations



$$u = -r\beta\sin\theta = -\beta y$$
$$v = r\beta\cos\theta = \beta x$$

$$\beta = \alpha z$$

α = *angle of twist per unit length*

$$u = -\alpha yz$$
$$v = \alpha xz$$
$$w = w(x, y)$$

w = *warping displacement*

Now must show assumed displacement form will satisfy all elasticity field equations

Stress Function Formulation

$$u = -\alpha yz \qquad v = \alpha xz \qquad w = w(x,y)$$

$$\Rightarrow \quad e_x = e_y = e_z = e_{xy} = 0, \quad e_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \alpha y\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \alpha x\right)$$

$$\Rightarrow \quad \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \quad \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} - \alpha y\right), \quad \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \alpha x\right)$$

Equilibrium Equations

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

Compatibility Relation

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -2\mu\alpha$$

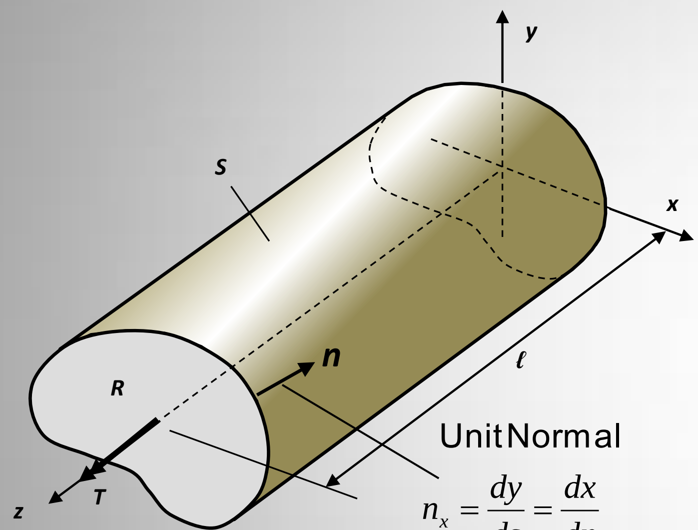
Introduce *Prandtl Stress Function* $\phi = \phi(x,y)$: $\tau_{xz} = \frac{\partial \phi}{\partial y}$, $\tau_{yz} = -\frac{\partial \phi}{\partial x}$

Equilibrium will be identically satisfied and compatibility relation gives

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\mu\alpha$$

a *Poisson equation* that is amenable to several analytical solution techniques

Boundary Conditions
Stress Function Formulation



Unit Normal

$$n_x = \frac{dy}{ds} = \frac{dx}{dn}$$

$$n_y = -\frac{dx}{ds} = \frac{dy}{dn}$$

<u>On Lateral Side: *S*</u>

$$T_x^n = \cancel{\sigma_x} n_x + \cancel{\tau_{yx}} n_y + \tau_{zx} \cancel{n_z} = 0 \Rightarrow 0 = 0$$

$$T_y^n = \cancel{\tau_{xy}} n_x + \cancel{\sigma_y} n_y + \tau_{zy} \cancel{n_z} = 0 \Rightarrow 0 = 0$$

$$T_z^n = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z \cancel{n_z} = 0 \Rightarrow$$

$$\frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} = 0 \Rightarrow \frac{d\phi}{ds} = 0 \Rightarrow \phi = \text{constant} = 0$$

<u>On End: *R (z = constant)*</u>

$$P_x = \iint_R T_x^n dxdy = 0$$

$$P_y = \iint_R T_y^n dxdy = 0$$

$$P_z = \iint_R T_z^n dxdy = 0$$

$$M_x = \iint_R y T_z^n dxdy = 0$$

$$M_y = \iint_R x T_z^n dxdy = 0$$

$$M_z = \iint_R (x T_y^n - y T_x^n) dxdy = T \Rightarrow T = 2\iint_R \phi \, dxdy$$

Displacement Formulation

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

Displacement component satisfies Laplace's equation

<u>On Lateral Side: *S*</u>

$$T_z^n = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = 0 \Rightarrow$$

$$\left(\frac{\partial w}{\partial x} - y\alpha\right) n_x + \left(\frac{\partial w}{\partial y} + x\alpha\right) n_y = 0 \text{ or } \frac{dw}{dn} = \alpha(y n_x - x n_y)$$

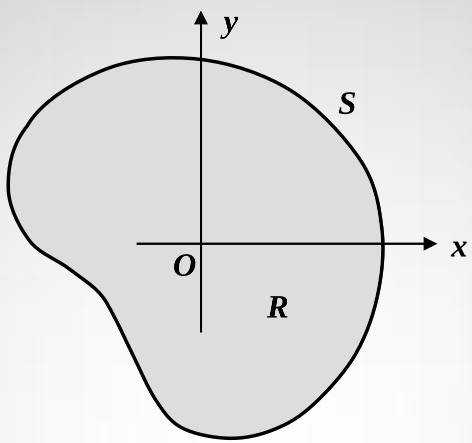
<u>On End: *R*</u>

$$M_z = \iint_R (x T_y^n - y T_x^n) dxdy = T \Rightarrow$$

$$T = \mu \iint_R \left(\alpha(x^2 + y^2) + x\frac{\partial w}{\partial y} - y\frac{\partial w}{\partial x} \right) dxdy$$

$$T = \alpha J \qquad J = \mu \iint_R \left(x^2 + y^2 + \frac{x}{\alpha}\frac{\partial w}{\partial y} - \frac{y}{\alpha}\frac{\partial w}{\partial x} \right) dxdy \ldots \text{ Torsional Rigidity}$$

Formulation Comparison



Stress Function Formulation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2\mu\alpha \in R$$

$$\phi = 0 \in S$$

Relatively Simple Governing Equation
Very Simple Boundary Condition

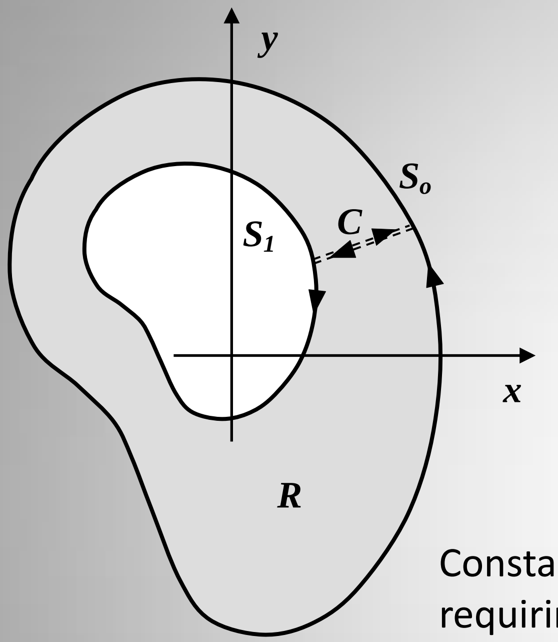
Displacement Formulation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \in R$$

$$\left(\frac{\partial w}{\partial x} - y\alpha\right)n_x + \left(\frac{\partial w}{\partial y} + x\alpha\right)n_y = 0 \in S$$

Very Simple Governing Equation
Complicated Boundary Condition

Multiply Connected Cross-Sections



Boundary conditions of zero tractions on all lateral surfaces apply to external boundary S_o and all internal boundaries S_1, . . . Stress function will be a constant and displacement be specified as per (9.3.20) or (9.3.21) on each boundary S_i, i = 0, 1, . . .

$$\phi = \phi_i \ \in S_i \quad \text{or} \quad \left(\frac{\partial w}{\partial x} - y\alpha\right)n_x + \left(\frac{\partial w}{\partial y} + x\alpha\right)n_y = 0 \ \in S_i$$

where ϕ_i are constants. Value of ϕ_i may be arbitrarily chosen only on one boundary, commonly taken as zero on S_o .
Constant stress function values on each interior boundary are found by requiring displacements w to be single-valued, expressed by

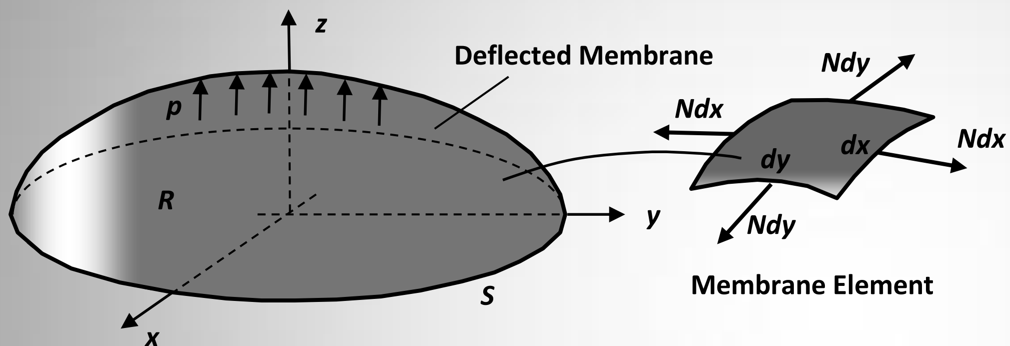
$$\oint_{S_1} dw(x,y) = 0 \quad \Rightarrow \quad \oint_{S_1} \tau ds = 2\mu\alpha A_1 \quad \text{where } A_1 \text{ is area enclosed by } S_1$$

Value of ϕ_1 on inner boundary S_1 must therefore be chosen so that relation is satisfied. If cross-section has more than one hole, relation must be satisfied for each hole.
Boundary conditions on cylinder ends will be satisfied, and resultant torque condition will give

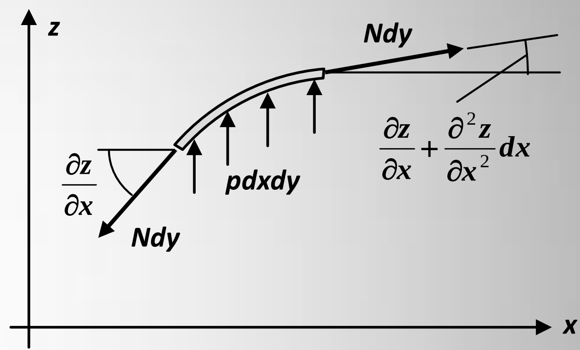
$$T = 2\iint_R \phi dxdy + 2\phi_1 A_1$$

Membrane Analogy

Stress function equations are identical to those governing static deflection of an elastic membrane under uniform pressure. This creates an *analogy* between the two problems, and enables particular features from membrane problem to be used to aid solution of torsion problem. Generally used to providing insight into qualitative features and to aid in developing approximate solutions.



Static Deflection of a Stretched Membrane



Equilibrium of Membrane Element

Membrane Equations

$$\sum F_z = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{N}$$

$$z = 0 \text{ on } S$$

$$V = \iint_R z\,dxdy$$

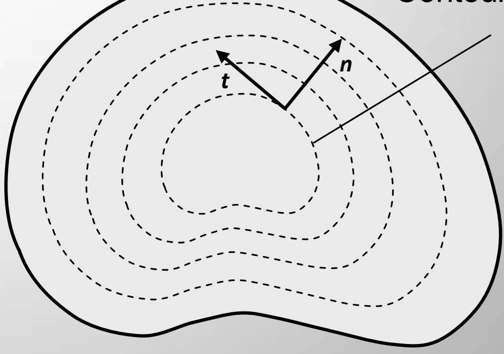
Torsion Equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\mu\alpha$$

$$\phi = 0 \text{ on } S$$

$$T = 2\iint_R \phi\,dxdy$$

Contour Line : z = constant

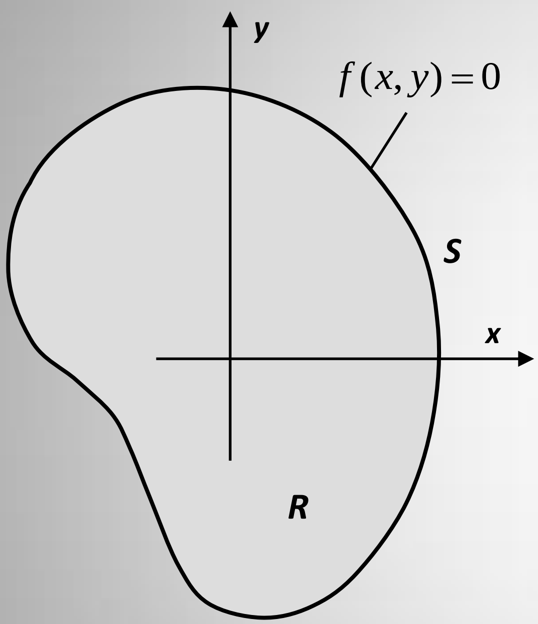


$$\Rightarrow \frac{\partial z}{\partial s} = 0 \Rightarrow \tau_{zn} = 0$$

$$\therefore \tau = \tau_{zt} = -\frac{dz}{dn}$$

Equations are same with: $\phi = z$, $p/N = 2\mu\alpha$, $T = 2V$

Torsion Solutions Derived from Boundary Equation



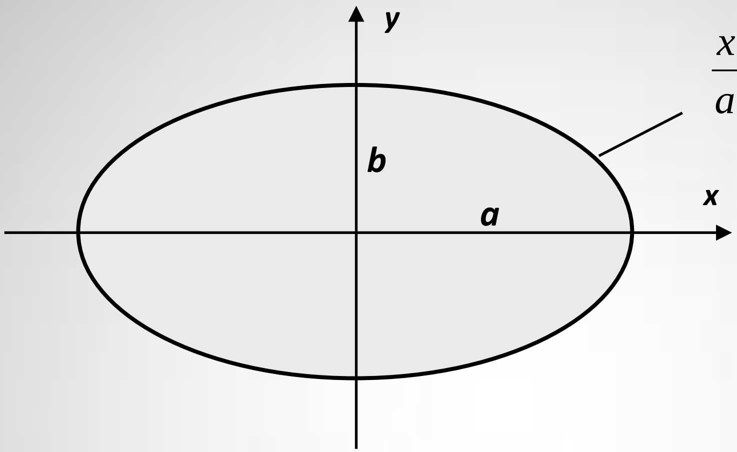
Boundary - Value Problem

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-2\mu\alpha \in R$$

$$\phi=0 \in S$$

If boundary is expressed by relation $f(x,y) = 0$, this suggests possible simple solution scheme of expressing stress function as $\phi = K\, f(x,y)$ where K is arbitrary constant. Form satisfies boundary condition on S, and for some simple geometric shapes it will also satisfy the governing equation with appropriate choice of K. Unfortunately this is not a general solution method and works only for special cross-sections of simple geometry.

Example 9.1 Elliptical Section



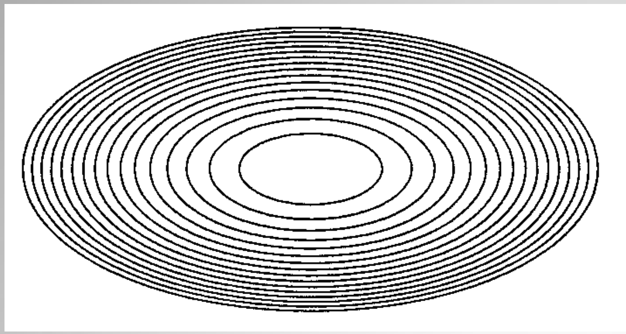
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Look for Stress Function Solution $\phi = K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

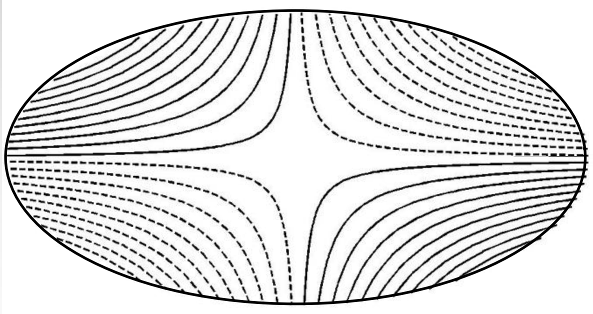
ϕ satisfies boundary condition and will satisfy governing governing if $K = -\frac{a^2 b^2 \mu\alpha}{a^2 + b^2}$

Since governing equation and boundary condition are satisfied, we have found solution

Elliptical Section Results



(Stress Function Contours)



(Displacement Contours)

Stress Field

$$\tau_{xz} = -\frac{2a^2\mu\alpha}{a^2+b^2}y = -\frac{2Ty}{\pi ab^3}$$

$$\tau_{yz} = \frac{2b^2\mu\alpha}{a^2+b^2}x = \frac{2Tx}{\pi ba^3}$$

$$\tau = \sqrt{\tau_{xz}^2+\tau_{yz}^2} = \frac{2T}{\pi ab}\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}}$$

$$\tau_{\max} = \tau(0,\pm b) = \frac{2T}{\pi ab^2}$$

Displacement Field

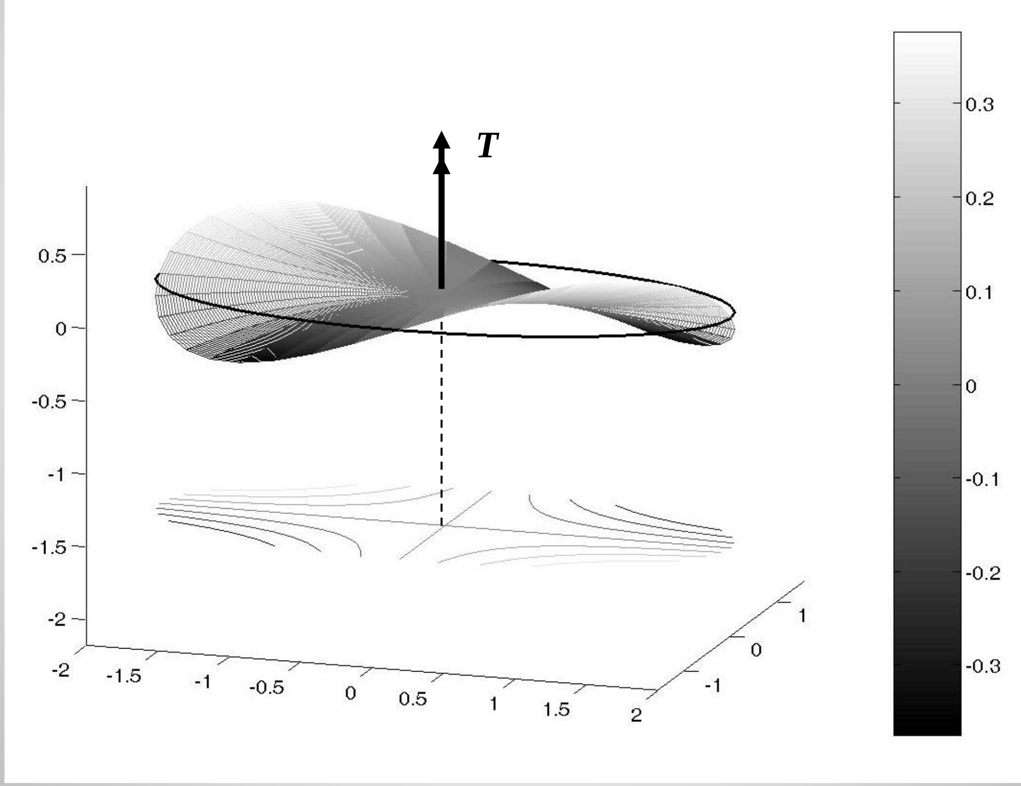
$$w = \frac{T(b^2-a^2)}{\pi a^3b^3\mu}xy$$

Loading Carrying Capacity
Angle of Twist

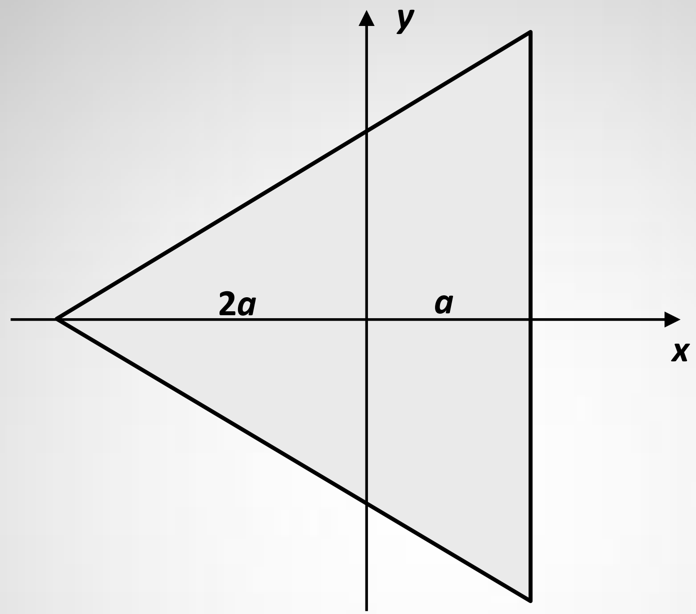
$$T = -\frac{2a^2b^2\mu\alpha}{a^2+b^2}\left(\frac{1}{a^2}\iint_R x^2dxdy+\frac{1}{b^2}\iint_R y^2dxdy-\iint_R dxdy\right)$$

$$T = \frac{\pi a^3b^3\mu\alpha}{a^2+b^2} \quad \text{or} \quad \alpha = \frac{T(a^2+b^2)}{\pi a^3b^3\mu}$$

Elliptical Section Results
3-D Warping Displacement Contours



Example 9.2 Equilateral Triangular Section



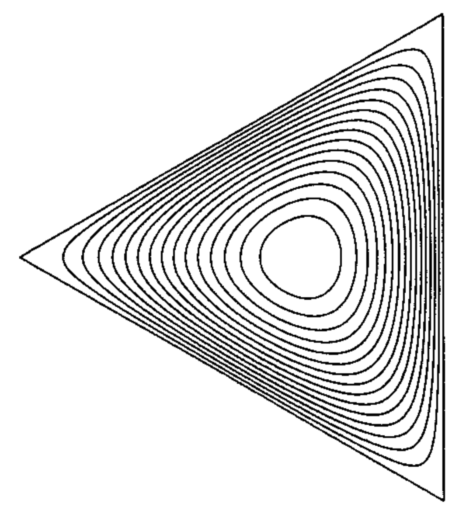
For stress function try product form of each boundary line equation

$$\phi = K(x - \sqrt{3}y + 2a)(x + \sqrt{3}y + 2a)(x - a)$$

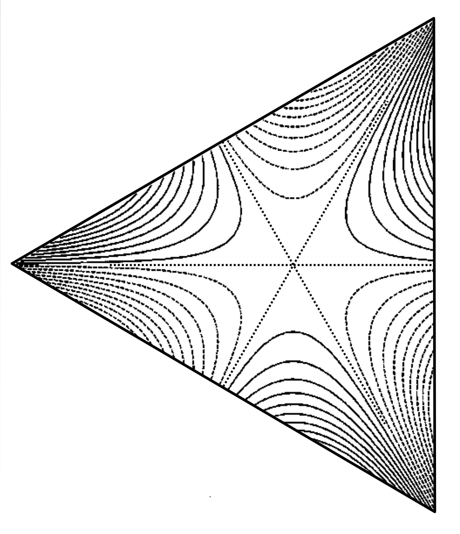
ϕ satisfies boundary condition and will satisfy governing governing if $K = -\frac{\mu\alpha}{6a}$

Since governing equation and boundary condition are satisfied, we have found solution

Equilateral Triangular Section Results



(Stress Function Contours)



(Displacement Contours)

Stress Field

$$\tau_{xz} = \frac{\mu\alpha}{a}(x-a)y$$

$$\tau_{yz} = \frac{\mu\alpha}{2a}(x^2 + 2ax - y^2)$$

$$\tau_{\max} = \tau_{yz}(a,0) = \frac{3}{2}\mu\alpha a = \frac{5\sqrt{3}T}{18a^3}$$

Displacement Field

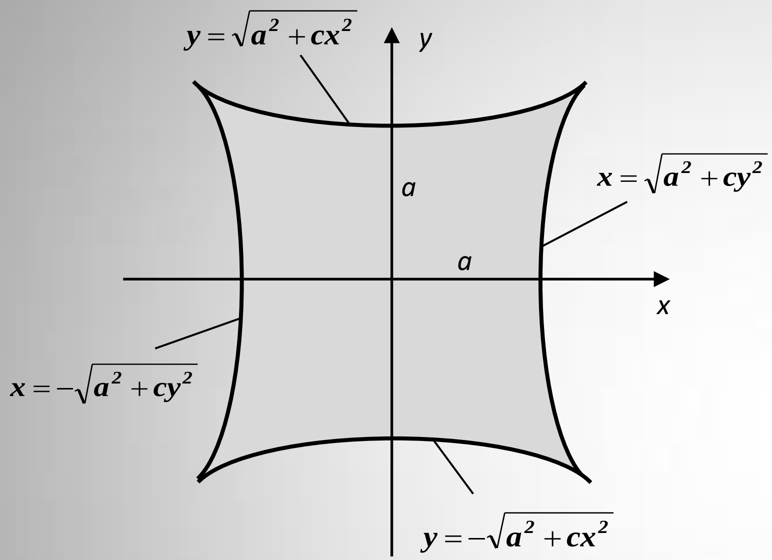
$$w = \frac{\alpha}{6a}y(3x^2 - y^2)$$

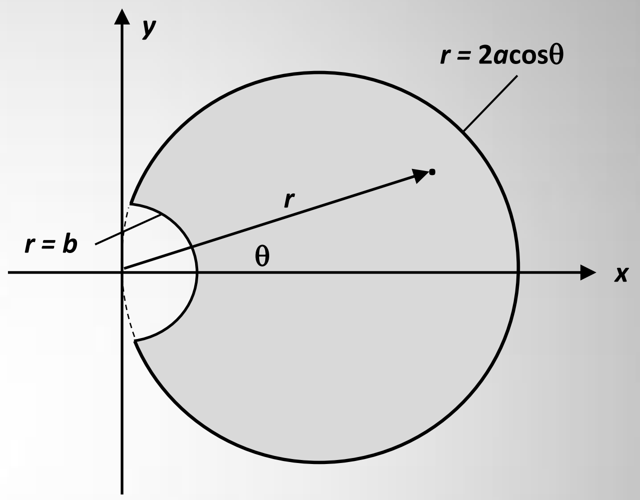
Loading Carrying Capacity

Angle of Twist

$$T = \frac{27}{5\sqrt{3}}\mu\alpha a^4 = \frac{3}{5}\mu\alpha I_p$$

Additional Examples That Allow Simple Solution Using Boundary Equation Scheme





Section with Higher Order Polynomial Boundary (Example 9-3)

$$\phi = K(a^2 - x^2 + cy^2)(a^2 + cx^2 - y^2)$$

$$K = -\frac{\mu\alpha}{4a^2(1-\sqrt{2})} \;,\; c = 3-\sqrt{8}$$

$$\tau_{\max} = \tau(\pm a,0) = \tau(0,\pm a) = \sqrt{2}\mu\alpha a$$

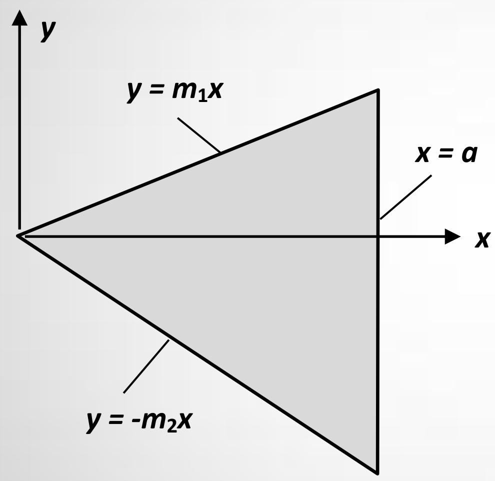
Circular Shaft with Circular Keyway (Exercise 9-22/23)

$$\phi = \frac{\mu\alpha}{2}(b^2 - r^2)(1 - \frac{2a\cos\theta}{r})$$

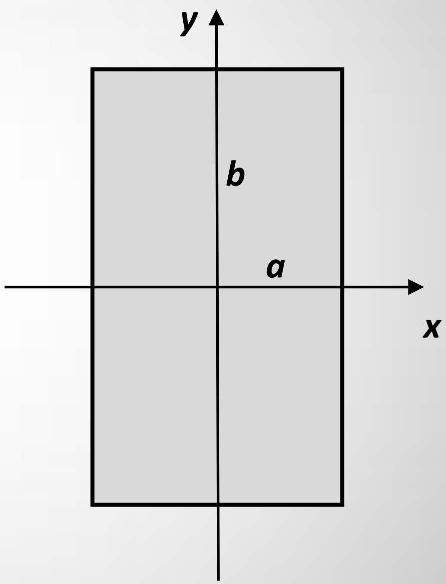
$$\text{As } b/a \to 0 \quad \frac{(\tau_{\max})_{keyway}}{(\tau_{\max})_{solidshaft}} \to \frac{2\mu\alpha a}{\mu\alpha a} = 2$$

$\therefore$ Stress Concentration of 2

Examples That Do Not Allow Simple Solution Using Boundary Equation Scheme

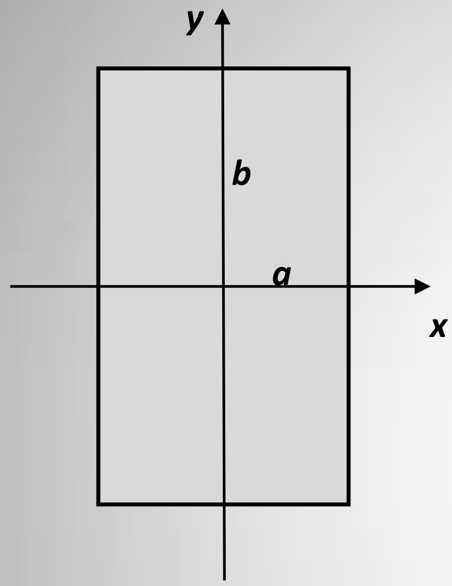


General Triangular Section



Rectangular Section

Example 9.4 Rectangular Section Fourier Method Solution



Previous boundary equation scheme will not create a stress function that satisfies the governing equation. Thus we must use a more fundamental solution technique - Fourier method. Thus look for stress function solution of the standard form

$$\phi = \phi_h + \phi_p \quad \text{with} \quad \phi_p(x,y) = \mu\alpha(a^2 - x^2)$$

homogeneous solution must then satisfy

$$\nabla^2\phi_h = 0\,,\ \ \phi_h(\pm a, y) = 0\,,\ \ \phi_h(x, \pm b) = -\mu\alpha(a^2 - x^2)$$

Separation of Variables Method ⇨ $\phi_h(x,y) = X(x)Y(y)$ ⇨ $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -2\mu\alpha$

⇨ $\phi_h(x,y) = \sum_{n=1}^{\infty} B_n \cos\dfrac{n\pi x}{2a}\cosh\dfrac{n\pi y}{2a}$ $\quad B_n = -32\mu\alpha a^2(-1)^{(n-1)/2} \Big/ \left(n^3\pi^3\cosh\dfrac{n\pi b}{2a}\right)$

$$\phi = \mu\alpha(a^2 - x^2) - \frac{32\mu\alpha a^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^3\cosh\frac{n\pi b}{2a}}\cos\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a}$$

Rectangular Section Results

Stress Field

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -\frac{16\mu\alpha a}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2\cosh\frac{n\pi b}{2a}}\cos\frac{n\pi x}{2a}\sinh\frac{n\pi y}{2a}$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = 2\mu\alpha x - \frac{16\mu\alpha a}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2\cosh\frac{n\pi b}{2a}}\sin\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a}$$

$$\tau_{\max} = \tau_{yz}(a,0) = 2\mu\alpha a - \frac{16\mu\alpha a}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2\cosh\frac{n\pi b}{2a}}$$

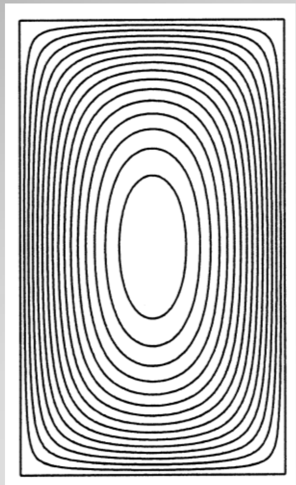
Loading Carrying Capacity/Angle of Twist

$$T = \frac{16\mu\alpha a^3 b}{3} - \frac{1024\mu\alpha a^4}{\pi^5}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}$$

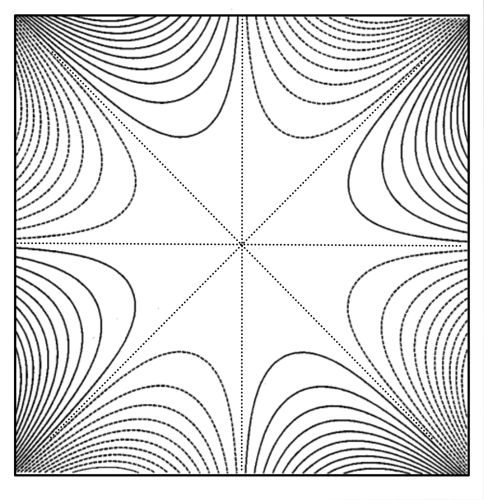
Displacement Field

$$w = \alpha xy - \frac{32\alpha a^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^3\cosh\frac{n\pi b}{2a}}\sin\frac{n\pi x}{2a}\sinh\frac{n\pi y}{2a}$$

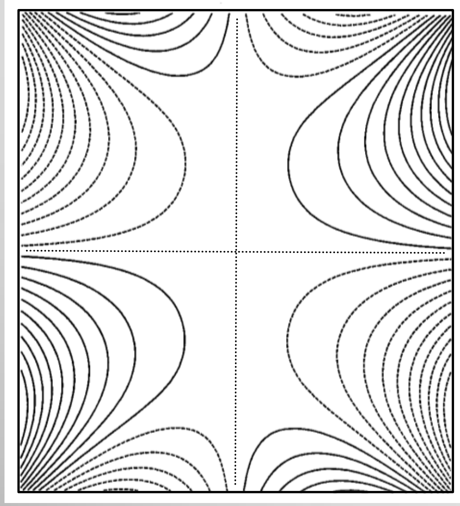
Rectangular Section Results



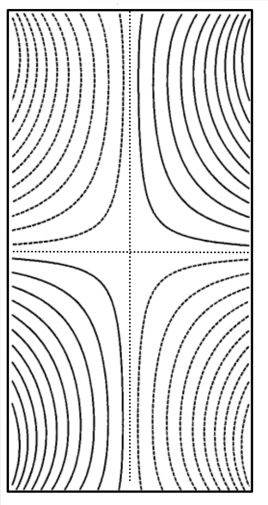
(Stress Function Contours)



(Displacement Contours, *a*/*b* = 1.0)

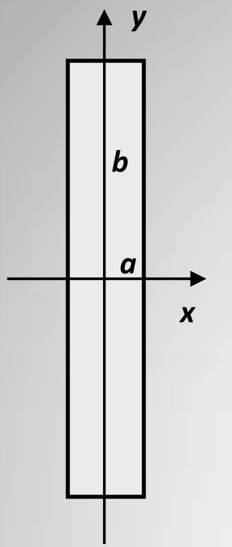


(Displacement Contours, *a*/*b* = 0.9)



(Displacement Contours, *a*/*b* = 0.5)

Torsion of Thin Rectangular Sections (a<<b)



Investigate results for special case of a *very thin rectangle* with $a << b$. Under conditions of $b/a >> 1$

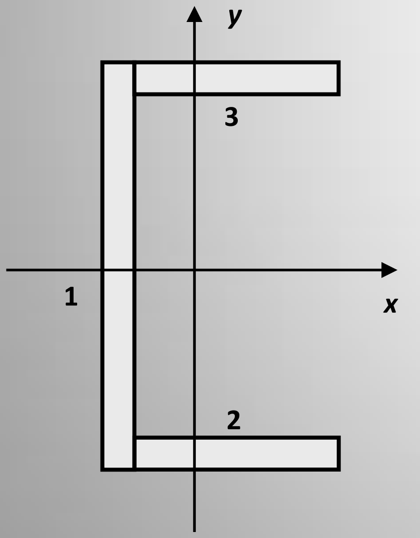
$$\cosh\frac{n\pi b}{2a} \to \infty \text{ and } \tanh\frac{n\pi b}{2a} \to 1$$

$$\phi = \mu\alpha(a^2 - x^2)$$

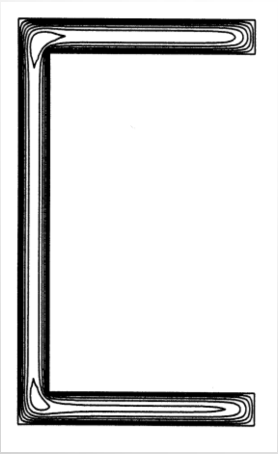
$$\tau_{\max} = 2\mu\alpha a$$

$$T = \frac{16}{3}\mu\alpha a^3 b$$

Composite Sections



(Composite Section)

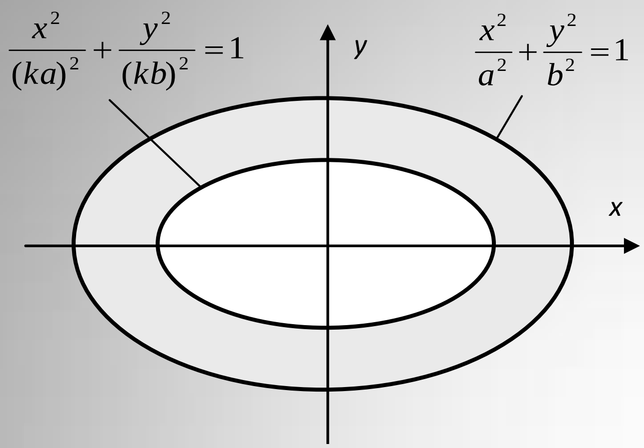


(Stress Function Contours)

Torsion of sections composed of thin rectangles. Neglecting local regions where rectangles are joined, we can use thin rectangular solution over each section. Stress function contours shown justify these assumptions. Thus load carrying torque for such composite section will be given by

$$T = \frac{16}{3}\mu\alpha\sum_{i=1}^{N} a_i^3 b_i$$

Example 9.5 Hollow Elliptical Section



For this case lines of constant shear stress coincide with both inner and outer boundaries, and so no stress will act on these lateral surfaces. Therefore, hollow section solution is found by simply removing inner core from solid solution. This gives same stress function and stress distribution in remaining material.

$$\phi=-\frac{a^2b^2\mu\alpha}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

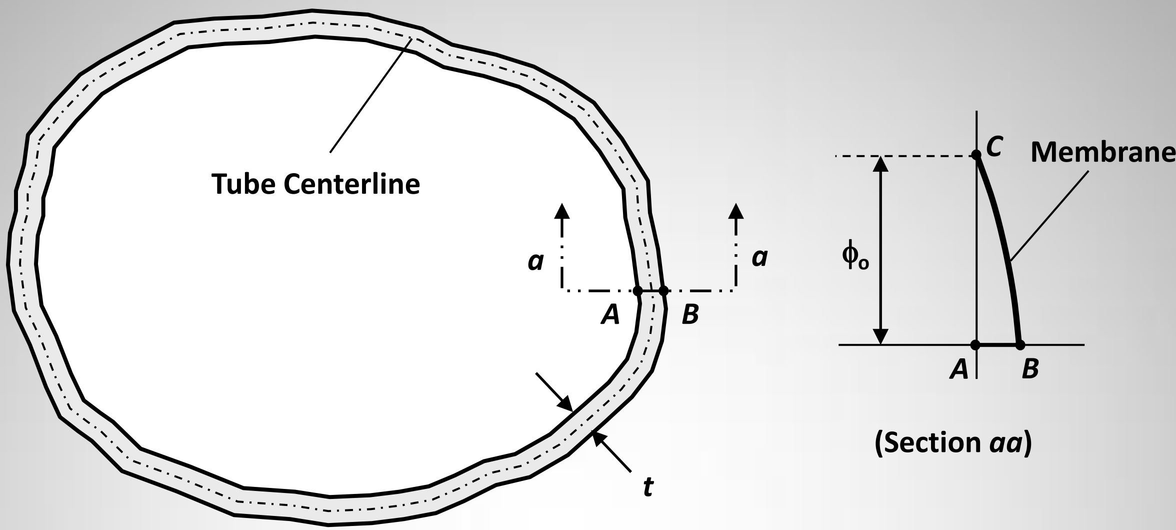
Constant value of stress function on inner boundary is $\phi_i=-\frac{a^2b^2\mu\alpha}{a^2+b^2}\left(k^2-1\right)$

Load carrying capacity is determined by subtracting load carried by the removed inner cylinder from the torque relation for solid section

$$T=\frac{\pi a^3b^3\mu\alpha}{a^2+b^2}-\frac{\pi(ka)^3(kb)^3\mu\alpha}{(ka)^2+(kb)^2}=\frac{\pi\mu\alpha}{a^2+b^2}a^3b^3(1-k^4)$$

Maximum stress still occurs at $x = 0$ and $y = \pm b$ $\quad\tau_{max}=\frac{2T}{\pi ab^2}\frac{1}{1-k^4}$

Hollow Thin-Walled Tube Sections



With t<<1 implies little variation in membrane slope, and *BC* can be approximated by a straight line. Since membrane slope equals resultant shear stress $\Rightarrow$ $\tau = \dfrac{\phi_o}{t}$

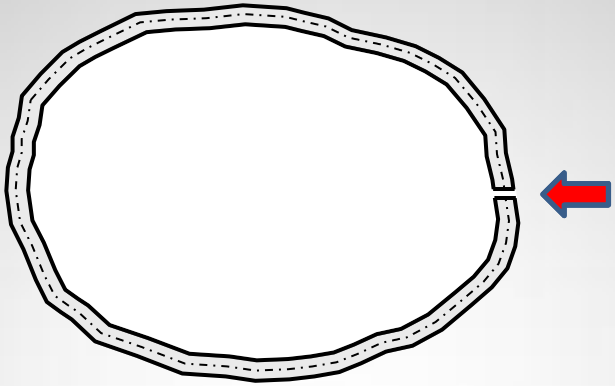
Load carrying relation: $T = 2\iint_R \phi\, dxdy + 2\phi_o A_i = 2\left(A\dfrac{\phi_o}{2}\right) + 2\phi_o A_i = 2\phi_o A_c$

where A = section area, A_i = area enclosed by inner boundary, A_c = area enclosed by centerline

Combining relations $\Rightarrow$ $\tau = \dfrac{T}{2A_c t}$

Angle of twist: $\oint_{S_c} \tau ds = 2\mu\alpha A_c \Rightarrow \alpha = \dfrac{TS_c}{4A_c^2 \mu t}$ where S_c = length of tube centerline

Cut Thin-Walled Tube Sections

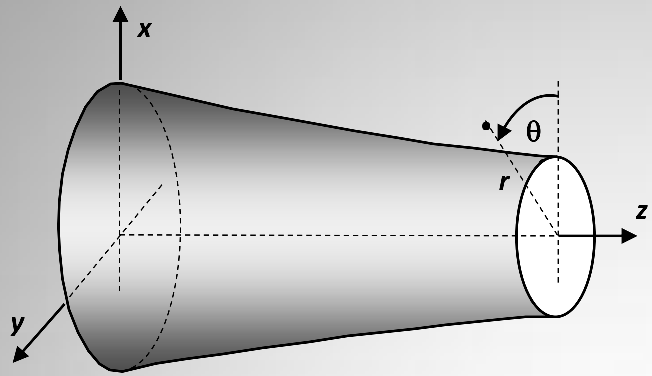


Cut creates an open tube and produces significant changes to stress function, stress field and load carrying capacity. Open tube solution can be approximately determined using results from thin rectangular solution. Stresses for open and closed tubes can be compared and for identical applied torques, the following relation can be established (see Exercise 9-24)

$$\frac{\tau_{OpenTube}}{\tau_{ClosedTube}} \approx \frac{\frac{3}{2}\frac{T}{aA_s}}{\frac{T}{2A_c t}} \approx 6\frac{A_c}{A_s}\text{, but since } A_c >> A_s \Rightarrow \frac{\tau_{OpenTube}}{\tau_{ClosedTube}} >> 1 \Rightarrow \tau_{OpenTube} >> \tau_{ClosedTube}$$

$$\therefore \text{Stresses are higher in open tube and thus closed tube is stronger}$$

Torsion of Circular Shafts of Variable Diameter



Displacement Assumption

$$u_r = u_z = 0$$
$$u_\theta = u_\theta(r,z)$$

$$e_r = e_\theta = e_z = e_{rz} = 0 \qquad \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),\ e_{\theta z} = \frac{1}{2}\frac{\partial u_\theta}{\partial z} \qquad \tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),\ \tau_{\theta z} = \mu\frac{\partial u_\theta}{\partial z}$$

Equilibrium Equations $$\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] + \frac{\partial}{\partial z}\left[r^3\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right)\right] = 0$$

Stress Function Approach ⇒

$$\frac{\partial \Psi}{\partial z} = -r^3\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = -\frac{r^2}{\mu}\tau_{r\theta}$$
$$\frac{\partial \Psi}{\partial r} = r^3\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right) = \frac{r^2}{\mu}\tau_{\theta z}$$

⇒ $$\frac{\partial^2\Psi}{\partial r^2} - \frac{3}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2\Psi}{\partial z^2} = 0$$

Boundary Condition

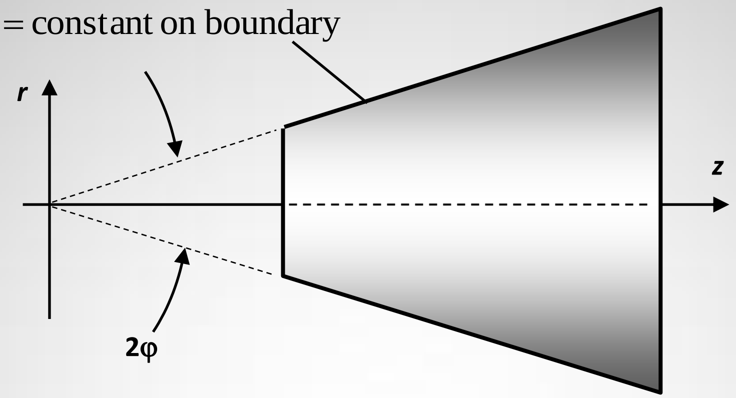
$$\frac{\mu}{r^2}\left(\frac{\partial\Psi}{\partial r}\frac{dr}{ds} + \frac{\partial\Psi}{\partial z}\frac{dz}{ds}\right) = 0 \Rightarrow \frac{d\Psi}{ds} = 0 \Rightarrow \Psi = \text{constant}$$

Load Carrying Torque

$$T = 2\pi\mu[\Psi(R(z),z) - \Psi(0,z)]$$

Conical Shaft Example 9-7

$$\frac{z}{\sqrt{r^2+z^2}} = \cos\varphi = \text{constant on boundary}$$



Stress Function Solution

$$\Psi = C\left(\frac{z}{\sqrt{r^2+z^2}} - \frac{1}{3}\frac{z^3}{(r^2+z^2)^{3/2}}\right) \qquad C = -\frac{T}{2\pi\mu(\frac{2}{3}-\cos\varphi+\frac{1}{3}\cos^3\varphi)}$$

Stresses

$$\tau_{r\theta} = -\frac{C\mu r^2}{(r^2+z^2)^{5/2}}$$

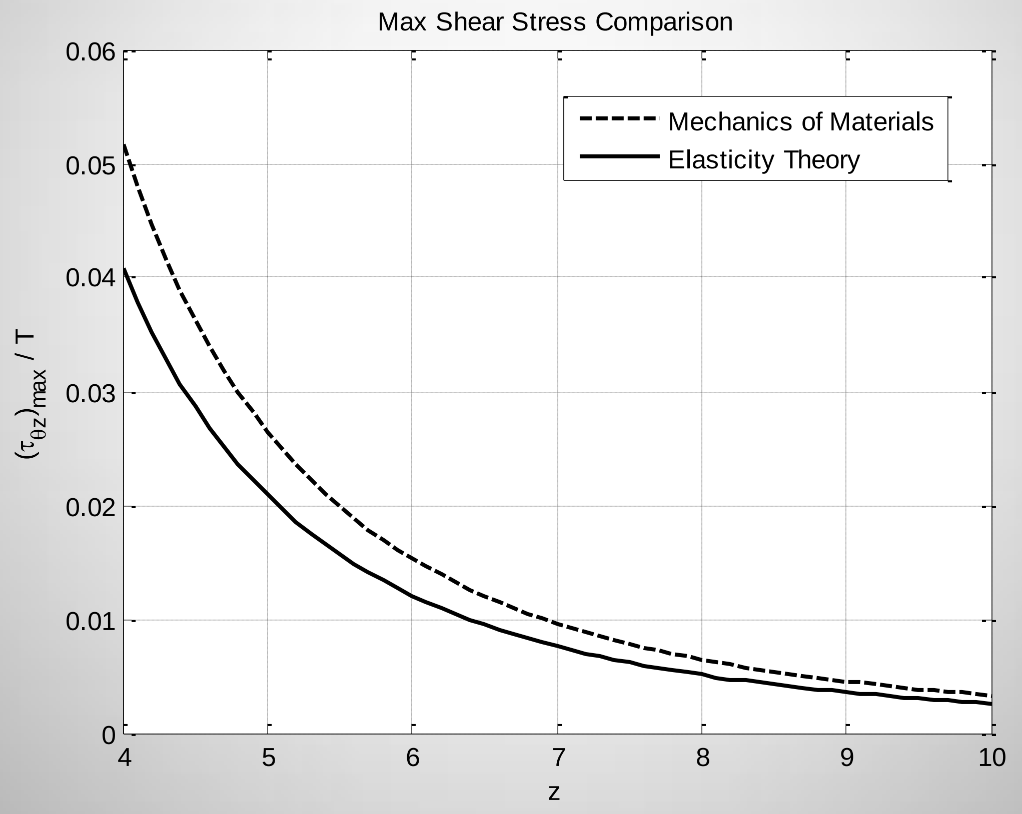
$$\tau_{\theta z} = -\frac{C\mu rz}{(r^2+z^2)^{5/2}}$$

Displacement

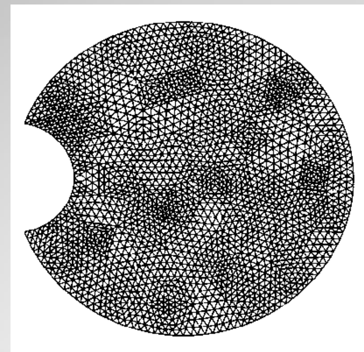
$$u_\theta = -\frac{Cr}{3(r^2+z^2)^{3/2}} + \omega r$$

ωr is rigid-body rotation about z-axis and ω can be determined by specifying shaft rotation at specific z-location

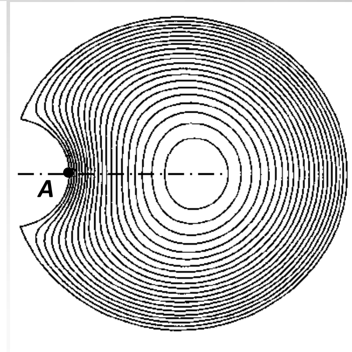
Conical Shaft Example 9-7 φ = 30º
Comparison with Mechanics of Materials



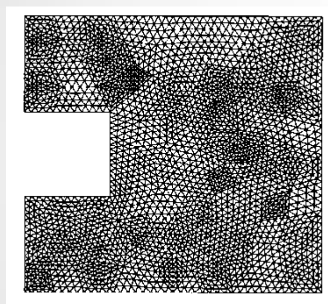
Numerical FEA Torsion Solutions



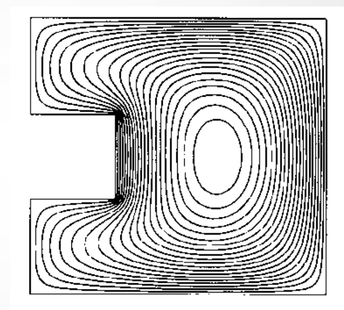
(4224 Elements, 2193 Nodes)



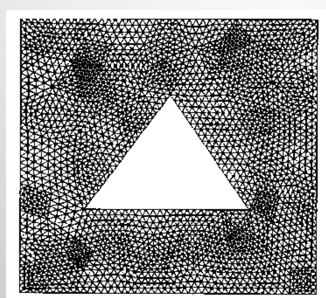
(Stress Function Contours)



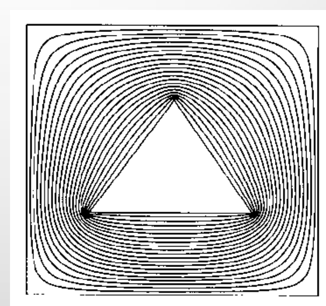
(4928 Elements, 2561 Nodes)



(Stress Function Contours)

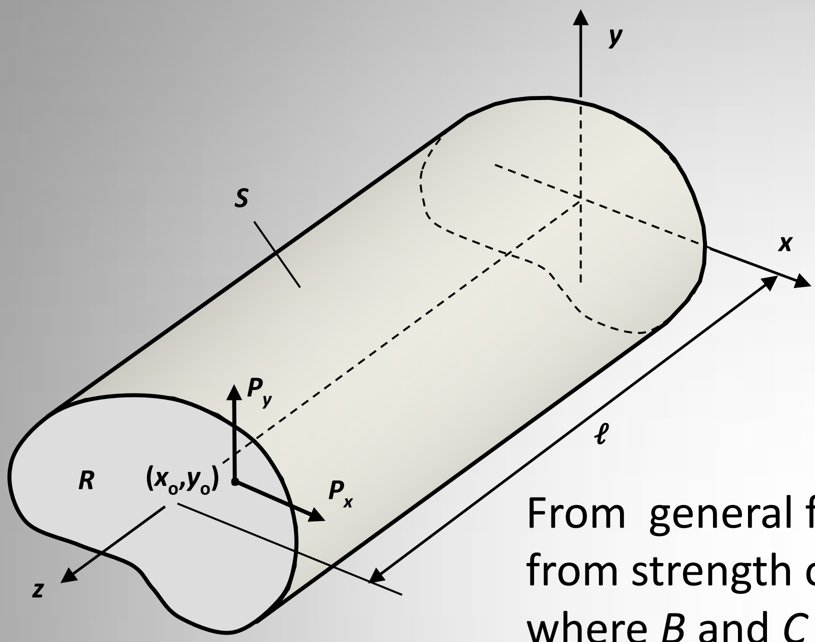


(4624 Elements, 2430 Nodes)



(Stress Function Contours)

Flexure of Cylinders



Consider flexure of cantilever beam of arbitrary section with fixed end at $z = 0$ and transverse end loadings P_x and P_y at $z = \ell$. Problem is solved in Saint-Venant sense, so only resultant end loadings P_x and P_y will be used to formulate boundary conditions at $z = \ell$.

From general formulation $\sigma_x = \sigma_y = \tau_{xy} = 0$, and motivated from strength of materials choose $\sigma_z = (Bx + Cy)(l - z)$, where B and C are constants. Stresses τ_{xz} and τ_{yz} will be determined to satisfy equilibrium and compatibility relations and all boundary conditions.

Remaining equilibrium equation $\dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} - (Bx + Cy) = 0$ will be identically

satisfied if we introduce stress function $F(x,y)$ such that

$$\tau_{xz} = \frac{\partial F}{\partial y} + \frac{1}{2}Bx^2$$

$$\tau_{yz} = -\frac{\partial F}{\partial x} + \frac{1}{2}Cy^2$$

Flexure Formulation

Remaining Beltrami-Michell Compatibility Relations

$$\frac{\partial}{\partial y}(\nabla^2 F) + \frac{\nu B}{1+\nu} = 0$$

$$-\frac{\partial}{\partial x}(\nabla^2 F) + \frac{\nu C}{1+\nu} = 0$$

$$\Rightarrow \quad \nabla^2 F = \frac{\nu}{1+\nu}(Cx - By) - 2\mu\alpha$$

Zero Loading Boundary Condition on Lateral Surface *S*

$$\tau_{xz} n_x + \tau_{yz} n_y = 0 \quad \Rightarrow \quad \frac{dF}{ds} = -\frac{1}{2}\left(Bx^2 \frac{dy}{ds} - Cy^2 \frac{dx}{ds}\right)$$

Separate Stress Function *F* into *Torsional Part* ϕ and *Flexural Part* ψ

$$F(x,y) = \phi(x,y) + \psi(x,y)$$

$$\nabla^2 \phi = -2\mu\alpha \text{ in } R \qquad \nabla^2 \psi = \frac{\nu}{1+\nu}(Cx - By) \text{ in } R$$

$$\frac{d\phi}{ds} = 0 \text{ on } S \qquad \frac{d\psi}{ds} = -\frac{1}{2}\left(Bx^2 \frac{dy}{ds} - Cy^2 \frac{dx}{ds}\right) \text{ on } S$$

Flexure Formulation

General solution to $\nabla^2\psi = \frac{\nu}{1+\nu}(Cx - By)$ ⇒

$$\psi(x,y) = f(x,y) + \frac{1}{6}\frac{\nu}{1+\nu}(Cx^3 - By^3) \text{ where } \nabla^2 f = 0$$

Boundary Conditions on end $z = \ell$

$$\iint_R \tau_{xz}\,dxdy = P_x \qquad \iint_R \tau_{yz}\,dxdy = P_y$$

⇒

$$BI_y + CI_{xy} = -P_x \qquad BI_{xy} + CI_x = -P_y$$

⇒

$$B = -\frac{P_x I_x - P_y I_{xy}}{I_x I_y - I_{xy}^2} \qquad C = -\frac{P_y I_y - P_x I_{xy}}{I_x I_y - I_{xy}^2}$$

where I_x, I_y and I_{xy} are the area moments of inertia of section R

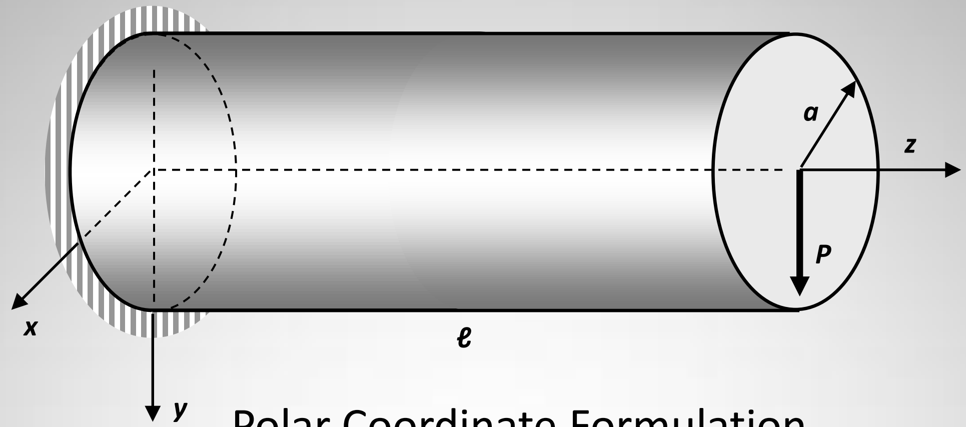
$$\iint_R [x\tau_{yz} - y\tau_{xz}]\,dxdy = x_o P_y - y_o P_x$$

⇒

$$\alpha J + \iint_R \left(\frac{1}{2}(Cxy^2 - Bx^2 y) - \left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right) \right) dxdy = x_o P_y - y_o P_x$$

where J is the torsional rigidity – final relation determines angle of twist α

Flexure Example - Circular Section with No Twist



Polar Coordinate Formulation

$$\nabla^2\psi = -\frac{\nu}{1+\nu}\frac{P}{I_x}r\cos\theta \qquad \frac{1}{a}\frac{\partial\psi}{\partial\theta} = \frac{1}{2}\frac{P}{I_x}a^2\sin^3\theta \quad \text{on } r=a$$

Solution: $\psi = \frac{P}{I_x}\left[-\frac{3+2\nu}{8(1+\nu)}a^2x - \frac{1+2\nu}{8(1+\nu)}xy^2 + \frac{1-2\nu}{24(1+\nu)}x^3\right]$

Stress Solution:

$$\tau_{xz} = -\frac{P}{4I_x}\frac{1+2\nu}{1+\nu}xy$$

$$\tau_{yz} = \frac{P}{I_x}\frac{3+2\nu}{8(1+\nu)}\left[a^2 - y^2 - \frac{1-2\nu}{3+2\nu}x^2\right] \Rightarrow \tau_{max} = \tau_{yz}(0,0) = \frac{P}{\pi a^2}\frac{3+2\nu}{2(1+\nu)}$$

$$\sigma_z = -\frac{P}{I_x}y(l-z)$$

Strength of Materials: $\tau_{max} = \frac{4}{3}\frac{P}{\pi a^2}$