Solving Poisson's equation using discontinuous elements with Comsol Multiphysics

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Poisson's equation

- Consider Poisson's equation:

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega, \\
u &= g \qquad \text{on } \partial\Omega.
\end{aligned}
$$

- In this example, we take

$$\Omega = (0,1)^2, \quad f = 2\pi^2 \sin(\pi x)\sin(\pi y), \quad g = 0,$$

  so that the exact solution is $u = \sin(\pi x)\sin(\pi y)$.

Poisson's equation

- The discontinuous Galerkin method is defined as finding $u_h \in W^h$ such that

$$\sum_{T\in\mathcal{T}_h} (\nabla u_h, \nabla v_h)_T - \sum_{e\in\mathcal{E}_h} \Big\{ \big\langle [\![u_h]\!] , \{\!\{\partial_n v_h\}\!\} \big\rangle_e + \big\langle \{\!\{\partial_n u_h\}\!\} , [\![v_h]\!] \big\rangle - \sigma h_e^{-1} \big\langle [\![u_h]\!] , [\![v_h]\!] \big\rangle_e \Big\} = (f, v_h) + \sum_{e\in\mathcal{E}_h^B} \big\langle g, \sigma h_e^{-1} v_h - \partial_n v_h \big\rangle_e .$$

- $\mathcal{E}_h$ - set of edges
- $\mathcal{E}_h^B \subset \mathcal{E}_h$- set of boundary edges

$$\begin{aligned} &\{\!\{\partial_n v\}\!\}\big|_e = \partial_{n_e} v^+ - \partial_{n_e} v^-, && [\![v]\!]\big|_e = v^+ - v^- && \text{if } e = \partial T^+ \cap \partial T^-, \\ &\{\!\{\partial_n v\}\!\}\big|_e = \partial_{n_e} v, && [\![v]\!]\big|_e = v && \text{if } e \in \mathcal{E}_h^B, \\ &(v^\pm = v\big|_{T^\pm}). \end{aligned}$$

- W^h denotes the space of discontinuous quadratic piecewise polynomials.
- $\sigma > 0$ is a stabilization parameter

Discontinuous Galerkin methods

- To implement in Comsol it is convenient to separate the interior edge terms and the boundary edge terms and move all of the terms to the left-hand side. That is, we write the method as

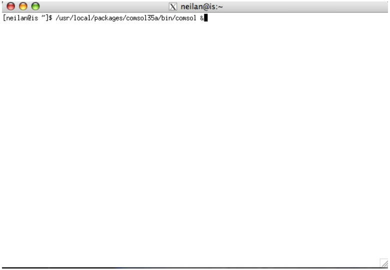
$$\sum_{T\in\mathcal{T}_h}\Big\{(\nabla u_h,\nabla v_h)_T-(f,v_h)_T\Big\}$$
$$-\sum_{e\in\mathcal{E}_h^I}\Big\{\big\langle [\![u_h]\!],\{\!\{\partial_n v_h\}\!\}\big\rangle_e+\big\langle\{\!\{\partial_n u_h\}\!\},[\![v_h]\!]\big\rangle-\sigma h_e^{-1}\big\langle[\![u_h]\!],[\![v_h]\!]\big\rangle_e\Big\}$$
$$-\sum_{e\in\mathcal{E}_h^B}\Big\{\big\langle\partial_n u_h,v_h\big\rangle_e+\big\langle u-g,\partial_n v_h-\sigma h_e^{-1}v_h\big\rangle_e\Big\}=0.$$

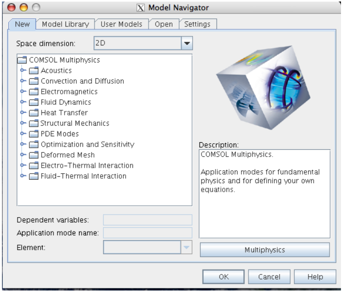
- $\mathcal{E}_h^I$ - interior edges

Step by step instructions

Step 1: Start the application Comsol Multiphysics. The path to the executable is

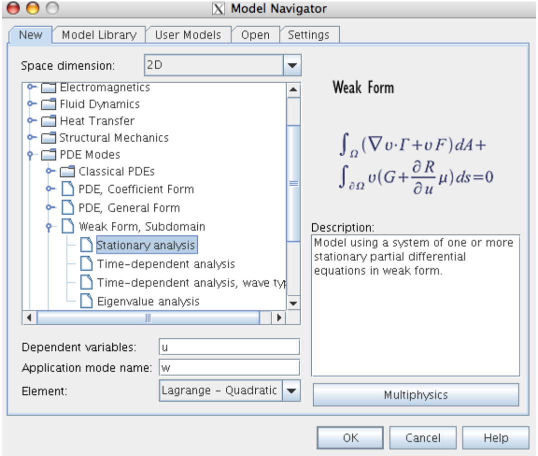
/usr/local/packages/comsol35a/bin/comsol





Step by step instructions

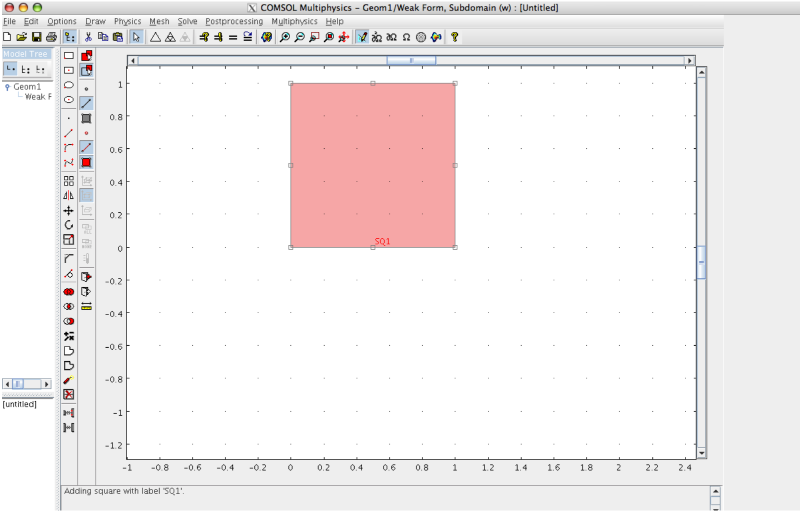
Step 2: (a) Select 'PDE Modes→Weak Form, Subdomain→ Stationary analysis'
(b) Select 'OK'



Step by step instructions

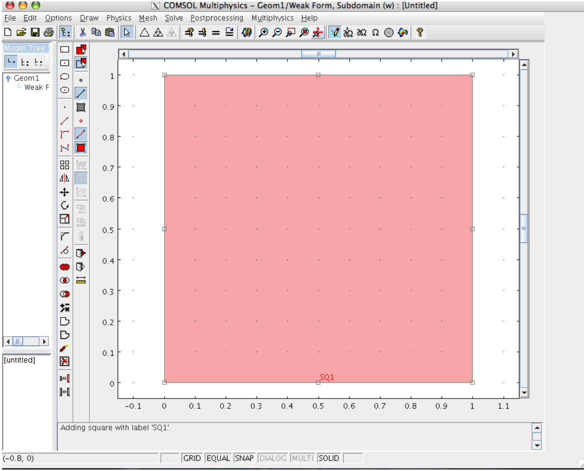
Step 3: (a) Select ‘Draw→Specific Object→Square’

(b) Select ‘OK’



Step by step instructions

Step 4: (optional) Click the icon for "Zoom Extents" - it is the icon with a magnifying glass and a red cross



Step by step instructions

Step 5: Select ‘Options→Global Expressions’

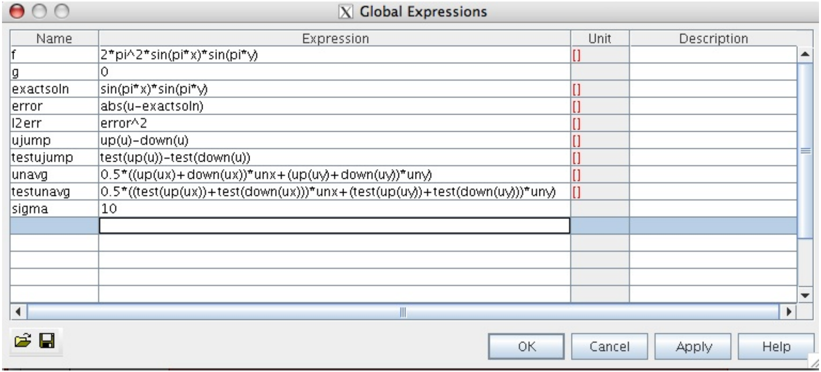
(a) Enter the following information in the given box:

Name	Expression	Unit	Description
f	2*pi^2*sin(pi*x)*sin(pi*y)		
g	0		
exactsoln	sin(pi*x)*sin(pi*y)		
error	abs(u-exactsoln)		
l2err	error^2		
ujump	up(u)-down(u)		
testujump	test(up(u))-test(down(u))		
unavg	0.5*((up(ux)+down(ux))*unx+(up(uy)+down(uy))*uny)		
testunavg	0.5*((test(up(ux))+test(down(ux)))*unx +(test(up(uy))+test(down(uy)))*uny)		
sigma	10		

(b) Select ‘Ok’

Step by step instructions

Step 5:



Name	Expression	Unit	Description
f	2*pi^2*sin(pi*x)*sin(pi*y)	[]	
g	0		
exactsoln	sin(pi*x)*sin(pi*y)	[]	
error	abs(u-exactsoln)	[]	
l2err	error^2	[]	
ujump	up(u)-down(u)	[]	
testujump	test(up(u))-test(down(u))	[]	
unavg	0.5*((up(ux)+down(ux))*unx+(up(uy)+down(uy))*uny)	[]	
testunavg	0.5*((test(up(ux))+test(down(ux)))*unx+(test(up(uy))+test(down(uy)))*uny)	[]	
sigma	10		

Step by step instructions

Step 6: (a) Select ‘Physics→Subdomain Settings’

(b) Enter the following information in each field:

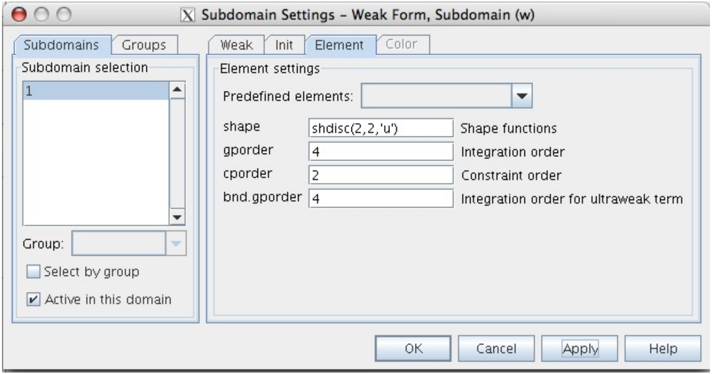
weak :	ux ∗ test(ux) + uy ∗ test(uy) − f ∗ test(u)
dweak :	0
bnd.weak :	−unavg ∗ testujump − ujump ∗ testunavg + (sigma/h) ∗ ujump ∗ testujump
constr :	0
Constraint type :	ideal
constrf :	0

(c) To use discontinuous elements, select the ‘Element tab’ and enter ‘shdisc(2,2,’u’)’ in the ‘shape’ field (the first 2 in ‘shdisc(2,2,’u’)’ is the dimension, and the second 2 is the polynomial degree).

(d) Select ‘Ok’

Step by step instructions

Step 6:



Step by step instructions

Step 7: (a) Select ‘Physics→Boundary Settings

(b) Under the “Weak” tab, select all four boundaries (1,2,3,4) and enter the following information:

weak : $-\text{test(u)} * (\text{ux} * \text{nx} + \text{uy} * \text{ny}) - \text{u} * (\text{test(ux)} * \text{nx} + \text{test(uy)} * \text{ny})$
$+(\text{sigma}/\text{h}) * \text{u} * \text{test(u)}$

dweak : 0

constr : 0

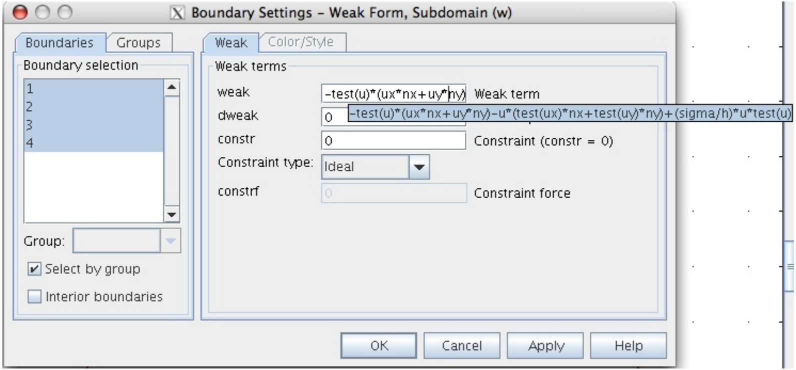
Constaint type : ideal

constrf : 0

(c) Select ‘OK’

Step by step instructions

Step 7:



Step by step instructions

Step 8: Select the ‘Solve’ icon (the icon with a plain equal sign)

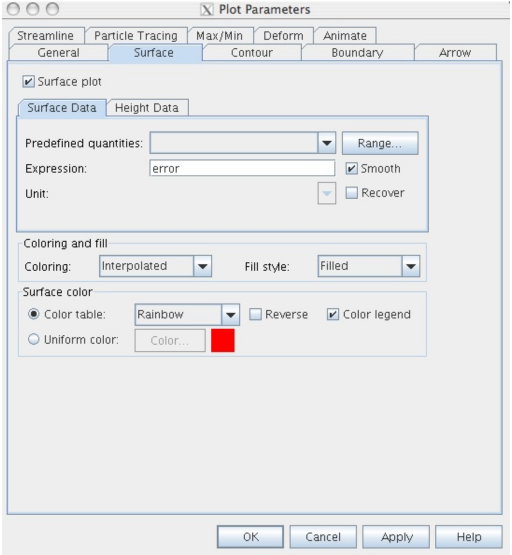
Step by step instructions

Step 9: To view the error,

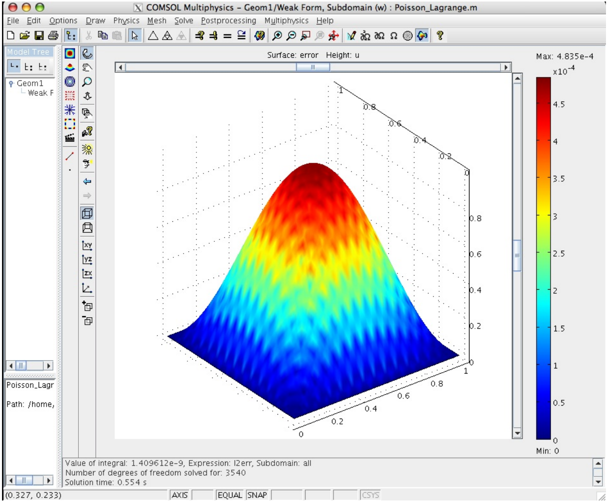
(a) Select 'Post Processing→Plot Parameters'
(b) Select the 'Surface tab'
(c) In the 'Surface Data Subtab', enter 'error' in the Expression field
(d) Select the 'Height Data Subtab' and check the box for Height Data
(e) Select 'Apply'

Step by step instructions

Step 10:



Step 10:


COMSOL Multiphysics - Geom1/Weak Form, Subdomain (w) : Poisson_Lagrange.m
File Edit Options Draw Physics Mesh Solve Postprocessing Multiphysics Help
Model Tree
Geom1
Weak F
Poisson_Lagr
Path: /home,
Surface: error Height: u
Max: 4.835e-4
x10^-4
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Min: 0
Value of integral: 1.409612e-9, Expression: l2err, Subdomain: all
Number of degrees of freedom solved for: 3540
Solution time: 0.554 s
(0.327, 0.233)
AXIS
EQUAL
SNAP
CSYS

Step by step instructions

Step 10: To calculate the error in the L^2 norm,

(a) Select 'Post Processing→ Subdomain Integration'

(b) Enter 'l2err' in the Expression field

(c) Select 'Apply'

(d) The value should appear in the lower left-hand side of the screen (note: this is the quantity $\|u - u_h\|_{L^2}^2$ *not* $\|u - u_h\|_{L^2}$).